

tected not only from its Patterson but also often from the photographs of its reciprocal lattice. The Patterson can be decomposed by means of a sublattice minimum function and manipulated so that the Patterson of the structure not concerned with the substructure is independently available for investigation.

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## ON DIFFRACTION BY A STRIP\*

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*Introduction.*—The problem of diffraction by an infinite strip or slit has been the subject of several investigations.<sup>1</sup> There are at least two "exact" methods for attacking this problem. One of these is the integral equation method,<sup>2</sup> the other the Fourier-Lamé method.<sup>3</sup> The integral equation obtained for this problem cannot be solved in closed form: expansion of the solution in powers of the ratio (strip width/wavelength) leads to useful formulas for low frequencies. In the Fourier-Lamé method the wave equation is separated in coordinates of the elliptic cylinder, the solution appears as an infinite series of Mathieu functions, and the usefulness of the result is limited by the convergence of these infinite series, and by the available tabulation of Mathieu functions.

The variational technique developed by Levine and Schwinger avoids some of the difficulties of the above-mentioned methods and, at least in principle, is capable of furnishing good approximations for all frequency-ranges. The scattered field may be represented as the effect of the current induced in the strip, and it has been proved by Levine and Schwinger<sup>4</sup> that it is possible to represent the amplitude of the far-zone scattered field in terms of the induced current in a form which is stationary with respect to small variations of the current about the true current. Substitution, in this representation, of a rough approximation for the current may give a remarkably good approximation of the far-zone scattered field amplitude. In this note we assume a normally incident field polarized parallel to the generators of the strip. As a rough approximation, we take a uniform density of the current induced in the strip. Since the incident magnetic field is constant over the strip, Fock's theory<sup>5</sup> may be cited in support of the uniformity of the current distribution, except near the edges where the behaviour of the field<sup>6</sup> indicates an infinite current density. A more detailed analysis of the current, by Moullin and Phillips,<sup>3</sup> is available but was not used here.

Once the (approximate) amplitude of the far-zone field has been obtained, the scattering cross-section may be found by the application of the scattering theorem<sup>4, 7, 8</sup> which relates this cross-section to the imaginary part of the amplitude of the far-zone scattered field along the central line of the umbral region. In spite of the crude approximation adopted for the induced current, the scattering cross-section shows a fair agreement with other available results.

*Integral Equation.*—We assume a plane wave with harmonic time dependence  $\exp(-i\omega t)$ , normally incident on the perfectly conducting infinitesimally thin strip of width  $2a$ ,

$$z = 0, \quad -a \leq y \leq a. \quad (1)$$

We further assume that the incident wave is polarized parallel to the edges of the strip (i.e., to the  $x$  axis) so that the only nonvanishing components of its electric field  $\mathbf{E}$ , magnetic field  $\mathbf{H}$ , and complex Poynting vector  $\mathbf{S}$  (all measured in MKS units, bars denoting conjugate complex numbers) are

$$E_x^i = \exp(ikz) \quad (2)$$

$$H_y^i = k/(\omega\mu) \exp(ikz) \quad (3)$$

$$S_z^i = E_x^i \overline{H_y^i} = k/(\omega\mu) \quad (4)$$

The scattered electric field is again parallel to the  $x$  axis, and the total electric field is the sum of the incident and scattered field,

$$E_x(y, z) = E_x^i(y, z) + E_x^{sc}(y, z) \quad (5)$$

The boundary condition on the screen is the vanishing of the electric field

$$E_x(y, 0) = 0 \quad \text{for } -a < y < a; \quad (6)$$

and the scattered field,  $E_x^{sc}$ , must represent, at large distances from the screen, an outgoing cylindrical wave (Sommerfeld's radiation condition).

The expression of the total electric field in terms of the induced current  $K_x(y)$  is

$$E_x(y, z) = \exp(ikz) - \omega\mu/4 \int_{-a}^a H_0^{(1)}(k[(y-y')^2 + z^2]^{1/2}) K_x(y') dy' \quad (7)$$

$i/4H_0^{(1)}(k[(y-y')^2 + z^2]^{1/2})$  being the two-dimensional free-space Green's function. The scattered field in (7) certainly satisfies Sommerfeld's radiation condition; in order that it also satisfies (6), we must have

$$1 = \omega\mu/4 \int_{-a}^a H_0^{(1)}(k|y-y'|) K_x(y') dy' \quad |y| \leq a \quad (8)$$

and this is the integral equation of our problem.

*Far-Zone Scattered Field Amplitude.*—We define the far-zone scattered field amplitude  $A(\phi)$  by

$$E_x^{sc} \sim i/4[2/(\pi i k \rho)]^{1/2} \exp(ik\rho) A(\phi) \quad k\rho \rightarrow \infty \quad (9)$$

where  $\rho \cos \phi = z$  and  $\rho \sin \phi = -y$ ; this expression represents a cylindrical outgoing wave of "amplitude"  $A$ .

Since

$$H_0^{(1)}(k[(y - y')^2 + z^2]^{1/2}) \sim [2/(\pi i k \rho)]^{1/2} \exp(ik\rho + iky' \sin \phi) \quad (10)$$

when  $k\rho \rightarrow \infty$ , we have from (7), for large  $k\rho$ ,

$$E_x^{sc} \sim i/4[2/(\pi i k \rho)]^{1/2} \exp(ik\rho) \int_{-a}^a i\omega\mu \exp(iky' \sin \phi) K_x(y') dy' \quad (11)$$

Comparing (9) and (11) we get

$$A(\phi) = i\omega\mu \int_{-a}^a \exp(iky' \sin \phi) K_x(y') dy' \quad (12)$$

*A Scattering Theorem.*—If  $P$  denotes the scattered energy flux per unit length, then the scattering cross-section  $\sigma$  is defined by

$$\sigma = P/S \quad (13)$$

where  $S = 1/2 \operatorname{Re} S_z^i = k/(2\omega\mu)$  is the incident energy flux per unit area.

$$P = 1/2 \operatorname{Re} \int_{-a}^a E_x^i(y', 0) \bar{K}_x(y') dy' - 1/2 \operatorname{Re} \int_{-a}^a K_x(y') dy' \quad (14)$$

since  $E_x^i = 1$  at  $z = 0$ . Hence,

$$\sigma = \omega\mu/k \operatorname{Re} \int_{-a}^a K_x(y') dy' \quad (15)$$

But from (12)

$$A(0) = i\omega\mu \int_{-a}^a K_x(y') dy' \quad (16)$$

Therefore, comparing (15) and (16) we get

$$\sigma = \operatorname{Im} A(0)/k \quad (17)$$

It may be remarked that, in the limit for very high frequencies, the cross-section (17) turns out to be twice the projected area. At first sight the factor two may seem inconsistent with geometrical optics, but actually the discrepancy is due to the different definitions of the scattering cross section.<sup>9</sup>

*Variational Principle.*—We multiply the integral equation (8) by  $K_x(y)$ , integrate from  $y = -a$  to  $y = +a$ , divide by the square of  $\int_{-a}^a K_x(y) dy$ , and then recall (16). Thus we obtain

$$A(0) = 4i \frac{\left[ \int_{-a}^a K_x(y) dy \right]^2}{\int_{-a}^a \int_{-a}^a K_x(y) H_0^{(1)}(k|y - y'|) K_x(y') dy dy'} \quad (18)$$

It is easy to show that the expression (18) is stationary with respect to small variations of  $K_x$  about the true  $K_x$  which satisfies the integral equation (8). Consequently, if we substitute a reasonable trial function for the unknown  $K_x$  in (18), we expect to obtain a reasonably accurate value of  $A(0)$ .

*Trial Function.*—According to Fock's theory<sup>5</sup> the current induced in the central region of the strip is equal to twice the tangential component of the incident magnetic field. And according to Bouwkamp,<sup>6</sup>  $H_y$  must have a singularity like  $(a^2 - y^2)^{-1/2}$  at the edges. This certainly conforms to the detailed information Moullin and Phillips<sup>3</sup> reported. In the present paper we take for a trial function  $K_x(y) = 1$ , thus ignoring the effect of the edges.

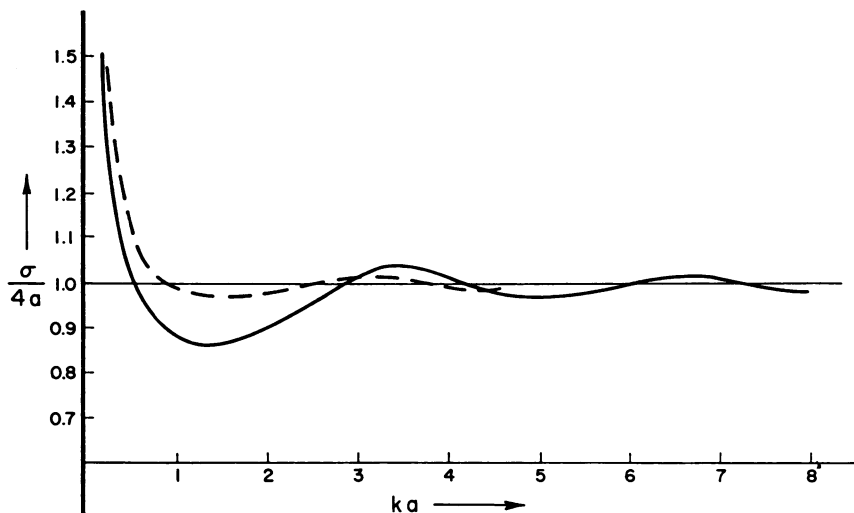


FIGURE 1

Plot of  $\sigma/(4a)$  versus  $ka$ . Solid curve is variational solution. Broken curve is Morse and Rubinstein's.

*Scattering Cross Section.*—With  $K_x(y) = 1$ , computation shows that (18) becomes

$$[A(0)]^{-1} = (4iak)^{-1} \left\{ \int_0^{2ka} H_0^{(1)}(t) dt - H_1^{(1)}(2ka) + (\pi ika)^{-1} \right\} \quad (19)$$

And applying the theorem (17) to (19), we get

$$\sigma/(4a) = \frac{\int_0^{2ka} J_0(t) dt - J_1(2ka)}{\left[ \int_0^{2ka} J_0(t) dt - J_1(2ka) \right]^2 + \left[ \int_0^{2ka} Y_0(t) dt - Y_1(2ka) - (\pi ka)^{-1} \right]^2} \quad (20)$$

The integrals appearing here have been tabulated.<sup>10</sup>

*Results.*—A plot of  $\sigma/(4a)$  versus  $ka$  is shown in the graph. This is a plot of equation (20). For small  $ka$  we have Rayleigh scattering. As  $ka$  increases the curve performs a damped oscillation about  $\sigma/(4a) = 1$ . And for  $ka \rightarrow \infty$  it can be shown by means of asymptotic representations of Bessel functions and their integrals that  $\sigma \rightarrow 4a$ .

According to Babinet's principle<sup>11</sup> the problem we have discussed is complementary to the problem of scattering by a slit for a normally incident wave polar-

ized perpendicular to the axis of the slit. For intermediate values of  $ka$  our curve behaves qualitatively as Morse and Rubinstein's:<sup>1</sup> the quantitative agreement is less good, the deviation being due to our choice of an overly simplified trial function. It is quite remarkable how such a rough approximation of the induced current yields fairly good results over the entire spectrum.

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## THE QUANTUM CORRECTION IN THE RADIATION BY ENERGETIC ACCELERATED ELECTRONS

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The radiation emitted by high energy electrons in a large scale magnetic field is an essentially classical phenomenon, and was so discussed in a paper<sup>1</sup> of the author. It was there remarked that the condition for quantum effects to be unimportant is that the momenta of the radiated quanta be small compared with the electron momentum. At very high electron energies, this requirement reads

$$\hbar\omega \ll E, \quad \omega \sim \omega_0(E/mc^2)^3, \quad (1)$$

where

$$\omega_0 = c/R = (eH/mc)(mc^2/E) \quad (2)$$

is the rotational frequency in the orbit determined by the magnetic field  $H$ . Hence